* Taylor Series Method for simultaneous first order differential equations.

The equation of the type $\frac{dy}{dx} = f_1(x, y, z)$

$\frac{dz}{dx} = f_2(x, y, z)$ with initial conditions.

$y(x_0) = y_0$, $z(x_0) = z_0$ can be solved by Taylor Series method as given below.

Problem

1). Solve $\frac{dy}{dx} = z - x$, $\frac{dz}{dx} = y + x$ with $y(0) = 1$, $z(0) = 1$, by taking $h = 0.1$, to get $y(0.1)$ and $z(0.1)$. Here y and z are dependent variables and x is independent.

Sol

$y' = z - x$ and $z' = x + y$

Take $x_0 = 0$, $y_0 = 1$

$y_1 = y(0.1) = ?$

$y' = z - x$

$y'' = z' - 1$

$y''' = z''$ etc

Take $x_0 = 0$, $z_0 = 1$ and $h = 0.1$

$z_1 = z(0.1) = ?$

$z' = x + y$

$z'' = 1 + y'$

$z''' = y''$ etc

By Taylor series, for y_1 and z_1, we have

$$y_1 = y(0.1) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \rightarrow ①$$

and $$z_1 = z(0.1) = z_0 + h z_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \cdots \rightarrow ②$$

$y_0 = 1$	$z_0 = 1$
$y_0' = z_0 - x_0 = 1 - 0 = 1$	$z_0' = x_0 + y_0 = 0 + 1 = 1$
$y_0'' = z_0' - 1 = 1 - 1 = 0$	$z_0'' = 1 + y_0' = 1 + 1 = 2$
$y_0''' = z_0'' = 2.$	$z_0''' = y_0'' = 0.$
	$z_0^{IV} = y_0''' = 2.$

Substituting these in ① and ②, we get

$$y_1 = y(0.1) = 1 + (0.1) + \frac{0.01}{2}(0) + \frac{0.001}{6}(2) + \cdots$$

$$= 1 + 0.1 + 0.000333$$

$$= 1.1003 \text{ (correct to 4 decimals)}$$

$$z_1 = z(0.1) = 1 + (0.1)1 + \frac{0.01}{2}(2) + \frac{0.001}{6}(0) + \frac{0.0001}{24}(2) + \cdots$$

$$= 1 + 0.1 + 0.01 + 0.0000083 + \cdots$$

$$= 1.1100 \text{ (correct to 4 decimal places)}$$

$\therefore$ $y(0.1) = 1.1003$ and $z(0.1) = 1.1100$.

* Taylor Series Method for second order differential equation

Any differential equation of the second or higher order can be solved by reducing it to a lower order differential equation.

A second order differential equation can be reduced to a first order differential equation by transformation $y' = z$ and then the latter one can be solved as usual.

Suppose $\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$

i.e., $y'' = f(x, y, y') \rightarrow (1)$

is the given differential equation together with the initial conditions

$y(x_0) = y_0 \rightarrow (2)$ and $y'(x_0) = y_0' \rightarrow (3)$

where y_0, y_0' are known values.

Setting $y' = p \longrightarrow (4)$

We get $y'' = p'$ and the equation (1) becomes

$p' = f(x, y, p) \rightarrow (5)$

with initial conditions,

$$y(x_0) = y_0 \rightarrow (6)$$

and $p(x_0) = p_0 = y_0' \rightarrow (7)$

Now, we resort to solve (5) together with (6) and (7) using Taylor series method.

$$p_1 = p_0 + hp_0' + \frac{h^2}{2!} p_0'' + \frac{h^3}{3!} p_0''' + \cdots \rightarrow (8)$$

where $p_1 = p(x = x_1)$ where $x_1 - x_0 = h$.

From (4),

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \cdots \text{ becomes}$$

$$y_1 = y_0 + p_0 + \frac{h^2}{2!} p_0' + \frac{h^3}{3!} p_0'' + \cdots \rightarrow (9)$$

Equation (5) gives p' and differentiating it, we get p'', p''', ... Hence p_0', p_0'', p_0''', ... can be got using (9) and (8) we can get y_1 and p_1. Since we know y_1, p_1 we can get p_1', p_1'', p_1''', ... at (x_1, y_1). Again using

$$p_2 = p_1 + hp_1' + \frac{h^2}{2!} p_1'' + \cdots \text{ we get } p_2 \text{ and using}$$

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \cdots \text{ we get } y_2$$

Since we calculate y_1', y_1'', y_1''', ... from (4)

Thus we calculate y_1, y_2, ...

Problem

27). Solve $y'' = y + xy'$ given $y(0) = 1$, $y'(0) = 0$ and calculate $y(0.1)$.

Sol. Here $x_0 = 0$, $y_0 = 1$, $y_0' = 0$.

$$y'' = y + xy'$$

Differentiating w.r.t x

$$y''' = y' + y' + xy'' = 2y' + xy''$$

$$y^{IV} = 2y'' + y'' + xy''' = 3y'' + xy'''$$

$$y^{V} = 4y''' + xy^{IV}$$

$$y^{VI} = 5y^{IV} + xy^{V}$$

$$y_0'' = y_0 + x_0 y_0' = 1$$

$$y_0''' = 2y_0' + x_0 y_0'' = 0$$

$$y_0^{IV} = 3y_0'' + x_0 y_0''' = 3$$

Here,

$$y(x) = y_0 + xy_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \cdots \quad y_0^{V} = 0;$$

$$y_0^{VI} = 15.$$

$$= 1 + 0 + \frac{x^2}{2}(1) + 0 + \frac{x^4}{4!}(3) + \cdots$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \cdots$$

$$\therefore y(0.1) = 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{8} + \frac{(0.1)^6}{48} + \cdots$$

$$= 1.00501252.$$

* Euler's Method

Aim : To solve $\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$ $\rightarrow$ ①

Let us take the points $x = x_0, x_1, x_2, \ldots$ where $x_i - x_{i-1} = h$,

i.e., $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots$

Let the actual solution of the differential equation be denoted by the graph (continuous line graph) $P_0(x_0, y_0)$. lies on the curve. We require the value of y on the curve at $x = x_1$. The equation of tangent at (x_0, y_0) to the curve is

$$y - y_0 = y'_{(x_0, y_0)}(x - x_0)$$

$$= f(x_0, y_0) \cdot (x - x_0)$$

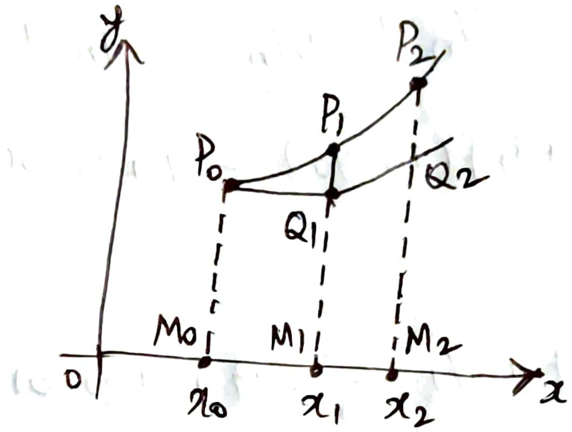
$$\therefore \quad y = y_0 + f(x_0, y_0)(x - x_0) \rightarrow ②$$

This y is the value of y on the tangent corresponding to $x = x$. In the interval (x_0, x_1), the curve is approximated by the tangent. Therefore, the value of y on the curve is approximately equal to the value of y on the tangent at (x_0, y_0) corresponding to $x = x_1$.

$\therefore \quad y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$

i.e., $y_1 = y_0 + hy_0'$, where $h = x_1 - x_0$.

$(M_1P_1 = M_1Q_1 = y_1)$ Refer figure.



$$\boxed{y(x+h) = y(x) + hf(x, y)}$$

Again, we approximate curve by the line through (x_1, y_1) and whose slope is $f(x_1, y_1)$ we get

$$y_2 = y_1 + hf(x_1, y_1) = y_1 + hy_1'$$

Thus

$$y_{n+1} = y_n + hf(x_n, y_n) \, ; \; n = 0, 1, 2, \ldots$$

This formula is called Euler's algorithms.

Problem

3). Given $y' = -y$ and $y(0) = 1$, determine the value of y at $x = (0.01)(0.01)(0.04)$ by Euler method.

Sol: $y' = -y$ and $y(0) = 1$; $f(x, y) = -y$.

Here, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.01$, $x_2 = 0.02$, $x_3 = 0.03$,

$x_4 = 0.04$.

We have to find y_1, y_2, y_3, y_4. Take $h = 0.01$.

By Euler algorithm,

$$y_{n+1} = y_n + h y_n' = y_n + h f(x_n, y_n). \rightarrow (1)$$

$$y_1 = y_0 + hf(x_0, y_0) = 1 + (0.01)(-1) = 1 - 0.01$$

$$= 0.99$$

$$y_2 = y_1 + h y_1' = 0.99 + (0.01)(-y_1)$$

$$= 0.99 + (0.01)(-0.99)$$

$$= 0.9801.$$

$$y_3 = y_2 + hf(x_2, y_2) = 0.9801 + (0.01)(-0.9801)$$

$$= 0.9703.$$

$$y_4 = y_3 + hf(x_3, y_3) = 0.9703 + (0.01)(-0.9703)$$

$$= 0.9606.$$

Tabular Values are

x	:	0	0.01	0.02	0.03	0.04
y	:	1	0.9900	0.9801	0.9703	0.9606
Exact y	:	1	0.9900	0.9802	0.9704	0.9608

since, $y = e^{-x}$ is the exact solution.